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Geometric Critical Exponent Inequalities for General Random Cluster Models

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A set of new critical exponent inequalities, $d(1-1/\delta) \geqslant 2-\eta$, $d\nu(1-1/\delta) \geqslant \gamma$, and $d\mu \geqslant 1$, is proved for a general class of random cluster models, which includes (independent or dependent) percolations, lattice animals (with any interactions), and various stochastic cluster growth models. The inequalities imply that the critical phenomena in the models are inevitably not mean-field-like in the dimensions one, two, and three.

KEY WORDS: Random cluster models; critical exponent inequalities; non-mean-field behavior.

1. INTRODUCTION

In the present paper[2] I derive a set of rigorous critical exponent inequalities for a general class of random cluster models. The class contains, in particular, (independent or dependent, site, bond, or site–bond) percolations, lattice animals (with any interactions), and various stochastic cluster growth models. The inequalities turn out to imply that the critical phenomena in the models in dimensions one, two, and three do not coincide with the mean-field theory predictions. Therefore one of their consequences is that the mean-field-type critical phenomena cannot be observed in any experiments of the random cluster models, provided that the experiments are done in our three-dimensional universe.

The proof of the inequalities is quite elementary and general. It is based only on the simple geometric fact that the "fractal dimension" of any geometric object is always not greater than the spatial dimension.

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The organization of the present paper is as follows. First I describe the random cluster models in a general setting, and then list some of the specific examples. Next I briefly describe my assumptions and main inequalities, and discuss their consequence. Finally I describe the precise assumptions and proofs of the inequalities.

2. RANDOM CLUSTER MODELS

Let Z^d be a d-dimensional *hypercubic lattice*, whose elements are called *sites*, and denoted by x, y,.... Denote the origin of Z^d by O. A *cluster* C is an arbitrary set of sites that contains O. Assume that, to each cluster C, there is associated an *appearance probability* $\text{Prob}_\beta(C)$, which satisfies the following normalization condition:

$$\sum_C \text{Prob}_\beta(C) = 1, \qquad \text{for all} \quad \beta \tag{1}$$

where the summation is over all the clusters. Here $\beta > 0$ is a physical parameter, which controls the nature of the model.

The basic physical quantities of the model are defined in the usual manner. The *cluster size distribution function* is defined by

$$P_n(\beta) = \sum_{C;|C|=n} \text{Prob}_\beta(C) \tag{2}$$

where the *size* of the cluster $|C|$ denotes the number of the sites contained in C. The *connectivity function* (or *correlation function*) $\tau_{ox}(\beta)$ is defined for any lattice site x by

$$\tau_{ox}(\beta) = \sum_{C;C\ni x} \text{Prob}_\beta(C) \tag{3}$$

Here the summation is over all the clusters containing x. Note that $\tau_{ox}(\beta)$ represents the probability that the cluster C contains the site x. The *mean cluster size* (or *susceptibility*) $\chi(\beta)$ and the *characteristic length* $\xi(\beta)$ are defined by

$$\chi(\beta) = \sum_C |C| \, \text{Prob}_\beta(C) = \sum_{n=1}^{\infty} nP_n(\beta) = \sum_{x \in Z^d} \tau_{ox}(\beta) \tag{4}$$

and

$$\xi(\beta) = \left(\frac{\sum_x |x|^\phi \, \tau_{ox}(\beta)}{\sum_x \tau_{ox}(\beta)} \right)^{1/\phi} \tag{5}$$

respectively. Here $\phi > 0$ is a certain fixed constant (say 2). Finally, I introduce the *connectivity function* under the *external magnetic field* $h > 0$ by

$$\tau_{ox}(\beta, h) = \sum_{C; C \ni x} \mathrm{Prob}_\beta(C)\, e^{-h|C|} \tag{6}$$

The corresponding *characteristic length* $\xi(\beta, h)$ is defined by (5), replacing $\tau_{ox}(\beta)$ by $\tau_{ox}(\beta, h)$.

Examples. Typical examples of such random cluster models are the following.

1. *General percolation models*[1]: In the (most general) site–bond percolation problem, each site x and each bond $\{x, y\}$ (x, y are two sites in Z^d, which need not be the nearest neighbors) are *occupied* or *unoccupied* according to some probabilistic rule. Any rules are allowed, including highly correlated ones, and it is assumed that the tendency of occupation increases as β increases. The most standard choice is the *independent percolation model*, where each site and bond are occupied with the independent probabilities $p_x = 1 - \exp(-\beta K_x)$ and $p_{xy} = 1 - \exp(-\beta J_{xy})$ (K_x and J_{xy} are fixed, nonnegative constants.) The above *cluster* C is defined as a set of the *occupied sites*, which are connected to the origin by the *occupied bonds*.

2. *General lattice animals*[2]: In the problems of lattice animals, one only considers a single cluster C including the origin, and specifies the appearance probabilities. A standard choice is to set

$$\mathrm{Prob}_\beta(C) = \frac{e^{-\beta|C|} f_\beta(C)}{\sum_C e^{-\beta|C|} f_\beta(C)} \tag{7}$$

where $f_\beta(C)$ is an arbitrary geometric factor. The basic *free lattice animal* is obtained by setting

$$f_\beta(C) = \begin{cases} 1 & \text{if } C \text{ is connected by the bonds of unit length} \\ 0 & \text{otherwise} \end{cases}$$

3. *Stochastic cluster growth models*[3]: The model is obtained by specifying a stochastic rule which determines the way of constructing a cluster $C(t+1)$, by adding some (or no) sites to the old cluster $C(t)$. Setting the initial condition as $C(0) = \{o\}$, and considering the $t \nearrow \infty$ limit of $C(t)$, one gets the above cluster C. Typical examples are the *epidemic model*[3] and its modifications. A set of models including DLA is not considered, since they do not have a power law cluster size distribution like (8).

3. CRITICAL PHENOMENA. ASSUMPTIONS

In a large class of the random cluster models, it is expected that there exist critical phenomena peculiar to the second-order phase transitions. At the *critical point* β_c, the following power law behavior of the cluster size distribution function and the connectivity function is expected:

$$P_n(\beta_c) \sim n^{-1/\delta - 1} \qquad \text{as} \quad n \nearrow \infty, \quad \delta > 1 \tag{8}$$

$$\tau_{ox}(\beta_c) \sim 1/|x|^{d-2+\eta} \qquad \text{as} \quad |x| \nearrow \infty \tag{9}$$

In the parameter region $\beta < \beta_c$, the cluster size distribution function $P_n(\beta)$ decays exponentially in n, and thus the quantities $\chi(\beta)$ and $\xi(\beta)$ are finite. But as β approaches β_c, both $\chi(\beta)$ and $\xi(\beta)$ are expected to diverge, exhibiting the following power law singularities:

$$\chi(\beta) \sim (\beta_c - \beta)^{-\gamma} \qquad \text{as} \quad \beta \nearrow \beta_c \tag{10}$$

$$\xi(\beta) \sim (\beta_c - \beta)^{-\nu} \qquad \text{as} \quad \beta \nearrow \beta_c \tag{11}$$

Finally, at β_c, the quantity $\xi(\beta_c, h)$ is also expected to show the power law singularity

$$\xi(\beta_c, h) \sim h^{-\mu} \qquad \text{as} \quad h \searrow 0 \tag{12}$$

4. NEW CRITICAL EXPONENT INEQUALITIES

The main result of the present paper is that, if one assumes the existence of the above critical phenomena, the critical exponents δ, η, γ, ν, and μ must satisfy the following three inequalities [actually, the precise assumptions needed for the proofs are weaker than (8)–(12); see below for the details]:

$$d(1 - 1/\delta) \geqslant 2 - \eta \tag{13}$$

$$d\nu(1 - 1/\delta) \geqslant \gamma \tag{14}$$

$$d\mu \geqslant 1 \tag{15}$$

Recall that, in the mean-field approximation[1] for the percolation models (and a certain class of the stochastic cluster growth models), these exponents are calculated as

$$\delta = 2, \qquad \eta = 0, \qquad \gamma = 1, \qquad \nu = 1/2, \qquad \mu = 1/4$$

and in the same approximation[2] for the lattice animals, these become

$$\delta = 2, \qquad \eta = 0, \qquad \gamma = 1/2, \qquad \nu = \mu = 1/4$$

Substituting each set of these mean-field values into the inequalities (13)–(15), one finds that each of the three inequalities implies $d \geqslant 4$. This fact means that the complete mean-field behavior is possible only in dimensions not less than four.

I make some remarks concerning these results.

1. In the language of the *critical dimensionality*, the inequalities (13)–(15) establish a rigorous lower bound $d_c \geqslant 4$. This bound is not optimal, since it is conjectured that $d_c = 6$ for the independent percolation models[1] and $d_c = 8$ for the free lattice animals.[2] See Ref. 4 for further results on the critical dimensions.

2. From the view point of the heuristic scaling theory,[1,2] the inequalities (13)–(15) are very natural. The scaling theory predicts the "scaling relations"

$$D(1 - 1/\delta) = 2 - \eta, \qquad D\nu(1 - 1/\delta) = \gamma, \qquad D\mu = 1$$

where D denotes the "fractal dimension" of a typical cluster at the critical point. The present inequalities are (heuristically) derived by combining the above scaling relations and a trivial bound $D \leqslant d$.

3. The inequalities (13)–(15) also hold for a model in a lattice other than the hypercubic lattice. In particular, one can treat any "fractal lattice" by replacing the dimension d by a suitable "fractal dimension" of the lattice.

4. In most lattice animals with $\mathrm{Prob}_\beta(C)$ of the form (7), the model is not defined in the region $\beta > \beta_c$ (since the summation in the denominator blows up). Sometimes, it also happens that the model is ill-defined even at the critical point $\beta = \beta_c$. Then the inequalities (13), (14) become meaningless, since the exponents δ and η are not defined. However, even in such a case, one can prove inequality (15) by slightly modifying the present method.

5. Combining (14) with Newman's inequality[5] $\gamma \geqslant 2(1 - 1/\delta)$ for the independent (site or bond) percolation models, one gets $d\nu \geqslant 2$. This inequality was recently obtained directly by Nguyen[6] and Chayes *et al.*[7]

5. PRECISE ASSUMPTIONS AND PROOFS

The precise assumptions needed for the proofs are as follows.

A. $\sum_{m=n}^{\infty} P_m(\beta_c) \leqslant C_1 n^{-1/\delta}$ holds for all $n = 0, 1, 2, \ldots$, with some $\delta > 1$ and $C_1 > 0$.

A′. $P_n(B_c) \geqslant C_2 n^{-1/\delta' - 1}$ holds for all $n = 0, 1, 2, \ldots$, with some $\delta' > 1$ and $C_2 > 0$.

B. $\sum_{x;|x| \leqslant L} \tau_{ox}(\beta_c) \geqslant C_3 L^{2-\eta}$ holds for all $L = 1, 2,...,$ with some $\eta < 2$ and $C_3 > 0$.

C. $\tau_{ox}(\beta) \leqslant \tau_{ox}(\beta_c)$ holds for all β such that $\beta_0 \leqslant \beta \leqslant \beta_c$, with some $\beta_0 < \beta_c$.

D. $\chi(\beta) \geqslant C_4(\beta_c - \beta)^{-\gamma}$ and $\xi(\beta) \leqslant C_5(\beta_c - \beta)^{-\nu}$ hold for all β such that $\beta_0 \leqslant \beta \leqslant \beta_c$, with some C_4, $C_5 > 0$.

E. $\xi(\beta_c, h) \leqslant C_6 h^{-\mu}$ holds for all h such that $0 \leqslant h \leqslant h_0$, with some $h_0 > 0$, $C_6 > 0$.

Note that all the assumptions A–E, except C, follow from the stronger assumptions[3] (8)–(12). The assumption C is very natural from our choice of the control parameter β.

Then the precise statements and their proofs are as follows.

Proposition 1. Assume A and B. Then we have the inequality $d(1 - 1/\delta) \geqslant 2 - \eta$.

Proof. Note that

$$\sum_{x;|x| \leqslant L} \tau_{ox}(\beta_c) = \sum_{C} |C \cap \Lambda_L| \operatorname{Prob}_{\beta_c}(C)$$

where Λ_L is the sublattice determined by the condition $|x| \leqslant L$. Dividing the summation according to the size of the cluster, we have

$$\leqslant \sum_{C;|C| \leqslant |\Lambda_L|} |C| \operatorname{Prob}_{\beta_c}(C) + \sum_{C;|C| > |\Lambda_L|} |\Lambda_L| \operatorname{Prob}_{\beta_c}(C) = \sum_{n=1}^{|\Lambda_L|} \sum_{m=n}^{\infty} P_m(\beta_c) \tag{16}$$

Here $|\Lambda_L|$ denotes the number of sites in Λ_L. Since $|\Lambda_L| \sim L^d$, we get the desired inequality from A and B. ∎

Proposition 2. Assume A, C, and D. Then we have the inequality $d\nu(1 - 1/\delta) \geqslant \gamma$.

Proof. Fisher[8] proved that

$$\chi(\beta) = \sum_{x} \tau_{ox}(\beta) \leqslant \text{const} \times \sum_{x;|x| \leqslant 2\xi(\beta)} \tau_{ox}(\beta)$$

[3] One can interpret the assumptions (8)–(10) as $\ln P_n(\beta_c)/\ln n \to -1/\delta - 1$, etc. Then the assumptions A–E are true if one replaces (for example) δ by $\delta + \varepsilon$ with arbitrary positive ε. Letting $\varepsilon \searrow 0$ after deriving the inequalities, one gets (13)–(15).

for any $\beta < \beta_c$. Then using the assumption C and the bound (16), we have

$$\chi(\beta) \leqslant \text{const} \times \sum_{x;|x| \leqslant 2\xi(\beta)} \tau_{ox}(\beta) \leqslant \sum_{n=1}^{|A_{2\xi(\beta)}|} \sum_{m=n}^{\infty} P_m(\beta_c)$$

Thus we get the desired inequality from A and D. ∎

Proposition 3. Assume A′ and E. Then we have the inequality $d\mu \geqslant (1 - 1/\delta')/(1 - 1/\delta)$. If $\delta = \delta'$, this reduces to the inequality (15).

Proof. Note that, in the region $0 \leqslant h \leqslant h_0$, we have

$$\sum_x \tau_{ox}(\beta_c, h) = \sum_{n=1}^{\infty} nP_n(\beta_c)\, e^{-hn} \geqslant \text{const} \times \sum_{n=1}^{1/h} nP_n(\beta_c)$$

Again, using the Fisher's bound and (16), we have

$$\sum_c \tau_{ox}(\beta_c, h) \leqslant \text{const} \times \sum_{x;|x| \leqslant 2\xi(\beta_c, h)} \tau_{ox}(\beta_c, h) \leqslant \text{const} \times \sum_{x;|x| \leqslant 2\xi(\beta_c, h)} \tau_{ox}(\beta_c)$$

Combining these two bounds, and using A, A′, we get

$$(1/h)^{1 - 1/\delta'} \leqslant \xi(\beta_c, h)^{d(1 - 1/\delta)}$$

Then the desired inequality follows from E. ∎

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